Option Pricing in Agricultural Markets

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This article provides an empirical application of the S. L. Heston and Nandi (2000) GARCH model as well as the S. L. Heston and Nandi (2000) option price valuation in aggricultural markets and compared to both the Black and Scholes (1973) model and option prices derived from a Monte Carlo simulation for different variance regimes. The analysis is based on the S&P GSCI spot price index for agricultural commodities. It turned out that the sample can be divided into a high and low volatility period with different variance risk premia. Moreover, the difference between option prices derived from the Black and Scholes (1973) option pricing framework and from the S. L. Heston and Nandi (2000) model heavily depends on the underlying volatility period, which is confirmed by different levels of moneyness as well as different times to maturity.

The appendix provides both a detailed derivation of the S. L. Heston and Nandi (2000) GARCH option valuation formula as well as R Codes for the estimation of the S. L. Heston and Nandi (2000) model via maximum likelihood techniques using BFGS algorithm.

Keywords: Agricultural commodity spot index, Heston-Nandi-GARCH, Heston-Nandi option pricing, Fourie transform, Monte Carlo Simulation, R-Code

1. Introduction and literature overview

Option valuation is a widely studied topic and many theoretical option pricing models were developed in the field of finance. The literature on option valuation starts with the Black and Scholes (1973) formula, which became a very useful tool in practical option pricing as well as the standard in the industry (Lee, Chen, and Lee (2016)). Nevertheless, a disadvantage is the fact that Black and Scholes (1973) assume a constant volatility of the underlying asset, which is empirically refuted, since it results in biases between market and model prices.

After the seminal work of Black and Scholes (1973), the literature grew fast to improve the problem with the wrong assumption of static volatiliy and came up with the idea of stochastic volatility, proving that biases could be reduced. There are two major categories of modeling stochastic volatility in option valuation: First, the continuous time approach of stochastic volatility and secondly the discrete time GARCH models. In continuous time stochastic volatility models variance is not directly observable and needs to be estimated seperately. In contrast, based on the seminal work of Engle (1982) and Bollerslev (1987) in discrete time stochastic volatility models, the conditional variance can be modeled directly within a GARCH framework.

Another distinction can be made between affine models versus non-affine models. The advantage of an affine structure is an almost closed form solution. The term *almost* comes from the fact that the models can be solved semi-analytically with numerical integration. In contrast, non-affine models must be solved by Monte Carlo simulations and are computationally not conveniant compared to affine models. The fact that these models are computationally expensive make them rather not feasible. Moreover, it could also be shown that it is very important to model a leverage effect that is the negative correlation between returns and volatility. For example, the S. Heston (1993) model, which belongs to continuous affine models, includes a leverage effect and was often used as a benchmark model in former studies.

A first discrete time option model was proposed by Duan (1995) with a non-affine GARCH model and based on this work, S. L. Heston and Nandi (2000) developed their affine GARCH model. Both models derive option prices by risk neutralization[1] through the local risk neutral valuation relationship (LRNVR). This valuation relationship results from Rubinstein (1976) and Brennan (1979) who constructed a framework to obtain a risk neutral valuation relationship (RNVR). Based on this work, Duan (1995) was the first who analysed this risk neutral framework for a GARCH model. Moreover, Duan (1995) constructed a model, which ensures the existence of a RNVR, he refers to as a LRNVR.

Christoffersen, Jacobs, and Mimouni (2006) compared the performance of both affine and non

[1] A risk-neutral probability measure is necessary to price options "at the expected discounted value of its future payoffs" (see Bates (2003)).

affine models in continuous time as well as in discrete time. Their most important finding is that the S. Heston (1993) model underperformes the S. L. Heston and Nandi (2000) model. Nevertheless, they also found that non-affine models perform better than affine models but they also came to the conclusion that a general comparison between continuous time and discrete time models would be misleading[2].

A necessary condition for applying a local risk neutral valuation relationship postulates a Gaussian underlying GARCH process. But when considering non-normal innovations, a LRNVR is not available. The literature uses two frameworks for pricing options with non-normal innovations: (1) an equilibrium approach and (2) an arbitrage free approach. In the first approach, Duan 1999 adjusted the LRNVR to the generalized LRNVR (GLRNVR). The second approach combines options and return data to estimate model parameters using a joint distribution. With this combination it is possible to archieve a connection between pyhsical measure P derived from the underlying generating process and the risk neutral measure Q derived from option prices. The link between the physical measure and the risk neutral measure is known as pricing kernel or stochastic discount factor (SDF), and its classical estimation is based on the Radon-Nikodym Theorem (see e.g. Jackwerth (2000)).

Obvisouly, the SDF is a monotonic decreasing function depending on the state of the economy[3]. Therefore, pricing kernels were initally constructed as a monotonic function of asset returns, which was at a later point of time critized because it could be shown that a monotonic setting is not always the case. Moreover, the literature provided evidence for increasing ranges of the pricing kernel and hence a non-monotonic function. This observation is now referred as pricing kernel puzzle (see e.g. Bakshi, Madan, and Panayotov (2010) or Linn, Shive, and Shumway (2017)) because it is economically counterintuitive[4].

As a direct result, Christoffersen, Heston, and Jacobs (2013) generalized the S. L. Heston and Nandi (2000) model by including a variance dependent pricing kernel, which seems to be able to answer the critique.

Simonato and Stentoft (2015) compared the performance of the above mentioned pricing approaches (1) and (2) and they concluded that both approaches lead to quite similar prices using

[2] Altough, a comparison between continuous and discrete models is difficult because the implementation is based on different econometric methods, Christoffersen et al. (2006) used an Auxiliary Particle Filter algorithm to make the two classes more comparable.

[3] The state of the economy is used to be modeled as the return on a broad market index and can be seen as aggregated wealth.

[4] This is because a marginal monetary unit would be appraised higher in times of rising markets than in falling markets.

time varying pricing parameters and the choice of which approach to use is up computational conveniance.

However, Linn et al. (2017) argued that the established pricing kernel estimators are the reason for the above mentioned pricing kernel puzzle because of inconsistency. They present a new nonparametric estimator and they found within a simulation analysis that their method outperforms existing classical estimation methods of the SDF.

However, the aim of this paper is not to fit given option data but rather value options in a potentially new market with quite different characteristics compared to conventional markets by comparing the approaches delivered by both Black and Scholes (1973) and S. L. Heston and Nandi (2000). The reason for this comparison is a practical one. On the one hand, the Black and Scholes (1973) model is the leading pricing model in practice and on the other hand, the S. L. Heston and Nandi (2000) model is relatively easy to implement and, because of its affine structure, convenient to handle. It will be shown that the pricing error between Black and Scholes (1973) and S. L. Heston and Nandi (2000) of option values will significantly depend on the state of volatility. In rather low volatiliy periods, the standard option model and the S. L. Heston and Nandi (2000) option model derive quite similar prices. But in high and rough volatility periods, the two models result in greater valuation differences. The two option prices resulting from Black and Scholes (1973) and S. L. Heston and Nandi (2000) are compared to option prices derived from a Monte Carlo anaylsis which serves as a benchmark. The latter confirms the observation that the price accuracy depends on the volatility period. Additionally, the dependence on the volatility state is also demonstrated by a rolling window analysis of option prices.

The paper is structured as follows: Section 2 describes the methodology as well as the data set used. This section derives the S. L. Heston and Nandi (2000) GARCH(1,1) model and option valuation formula in detail. The third part investigates the model results and pricing differences in different volatility regimes. The paper ends with a conclusion and a summary of the results. Moreover, the appendix provides the R Code for the estimation process as well as the option valuation.

2. Description of the dataset used and preliminary analysis

In order to investigate option prices in agricultural markets, I perfomed the analysis with the S&P GSCI Agriculture Spot Index. The index data were downloaded from Thomson Reuters Datastream[5] for the time period 01.01.2008 to 31.12.2016. The dataset contains 2,349 daily observations, which results in 2,348 daily log returns calculated as $r_t = log(S_t) - log(S_{t-1})$ where S_t is the spot price at time t.

The index is a widely used benchmark for investors in agricultural commodity markets and is quite comfortable to be used as an underlying process for agricultural markets. In figure 1 (upper

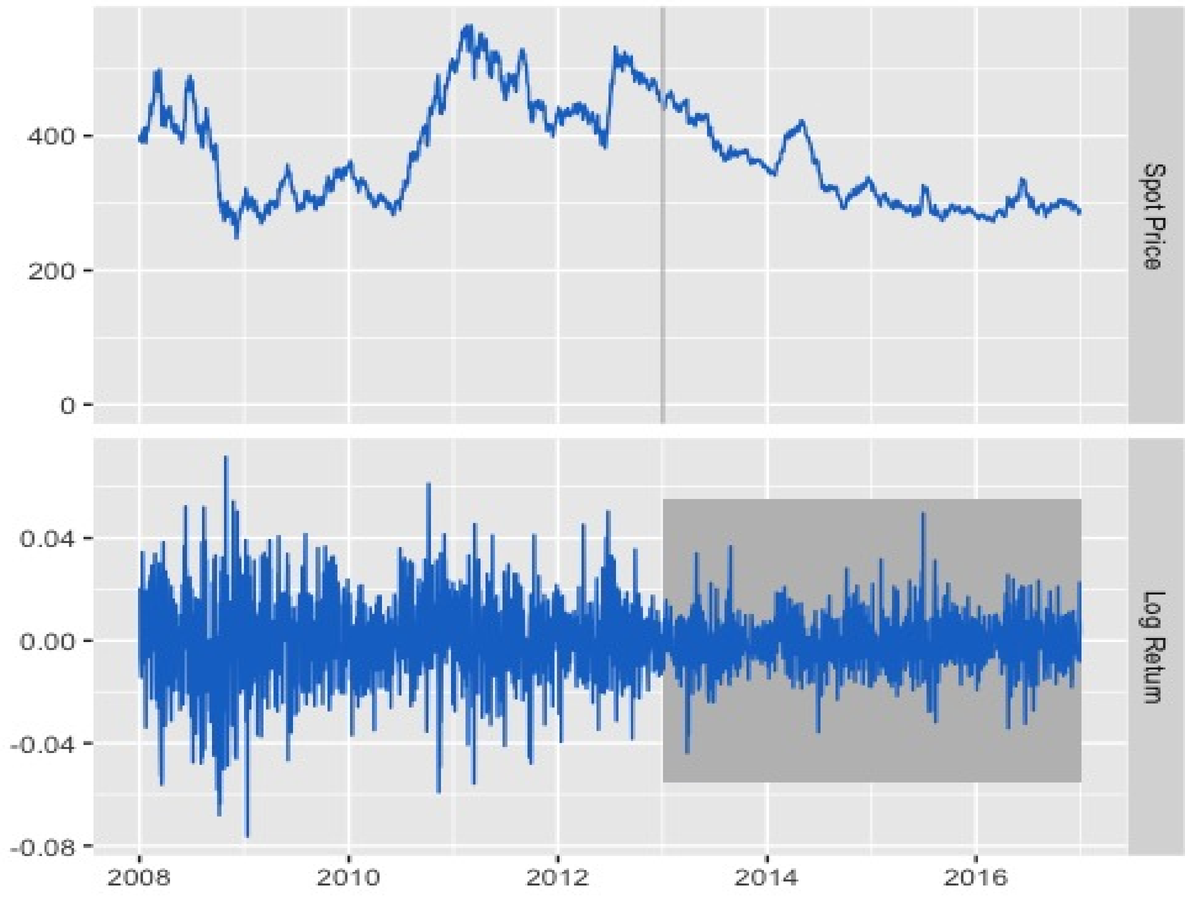

Figure 1: Spot prices and log returns of S&P GSCI Agriculture Spot Index from 2008 to 2016

part) the index clearly shows the food price crises in 2008, 2011, and 2012. Starting with 2012, food prices relax and it seems that they have been on a stable level since 2015.

Looking at the return series of the index (lower part), it is recognizable that during the financial crisis in 2008/2009 and the food crisis 2011 volatility clustering exists. Additionally, figure 1 shows that the decrease of volatility clustering coincidences with decreasing food prices since 2012

[5] Datastream RIC is $.SPGSAG$

(a very hot and dry summer was responsible for this price peak). Therefore, one can suggest a break point in 2012. To manifest this, I applied a non-parametric change point analysis as can be found in N Ross and Adams (2012). Within this analysis, I estimated the change point with the Cramer-von-Mises test, which can be used to detect changes in a framework with non-Gaussian distributions[6]. The test detected the change point on 20.8.2012 as already suggested. Therefore, I split the sample into two sub-samples. The first sub-sample embraces a time span from 1.1.2008 until 20.8.2012 and the second sub-sample goes from 21.8.2012 until 31.12.2016. The vertical line in the upper part of figure 1 and the shadowed area in the lower part, respectively, mark the two sub-samples.

Table 1 summarizes the descriptive statistics of the returns for the sample as well as for the chosen subsamples.

Panel 1: Whole sample period from 1.1.2008 - 31.12.2016

mean	std. dev.	skewness	exc. kurtosis	min	max	1st Qu.	med.	3rd Qu.
-0.00012	0.0142	-0.1491	2.5057	-0.0763	0.0715	-0.0076	0.0000	0.0074

Panel 2: Sub-sample period from 1.1.2008 - 20.8.2012

mean	std. dev.	skewness	exc. kurtosis	min	max	1st Qu.	med.	3rd Qu.
0.00025	0.0173	-0.2131	1.3694	-0.0763	0.0715	-0.0094	0.0000	0.0107

Panel 3: Sub-sample period from 21.8.2012 - 31.12.2016

mean	std. dev.	skewness	exc. kurtosis	min	max	1st Qu.	med.	3rd Qu.
-0.00052	0.0098	0.0681	1.6381	-0.0476	0.0496	-0.0067	-0.0003	0.0057

Table 1: Descriptive statistics of the sample and sub-samples

Panel 1 of table 1 reports the descriptive results for the complete period. We can observe a negative unconditional mean as well as a negative skewed and heavy tailed distribution. Panel 2 prints the results for the first period until 20.8.2012. In this time span, we see a positive unconditional mean of the returns, but in turn a more negative skewed and less heavy tailed distribution compared to the entire sample data. Panel 3 shows the values for the second sub-sample. The mean return is negative but the skewness shows a positive sign and, compared to Panel 2, higher excess kurtosis.

[6] Before I started the analysis of the historical S&P data I simulated GARCH processes with different parameters to receive a series with evident break points. Obviously the overall assumption of this simulation is that the historical return series follows a GARCH process as well. I tested both the Kolmogorov-Smirnov test and the Cramer-von-Mises test and came to the conclusion that the latter performed better. This is also documented in ?.

This result is quite remarkable because financial time series data use to exhibit both negative skewness and a positive variance risk premium. Moreover, the information is important for the continuing analysis in the upcoming sections.

3. Description of the methodology

3.1. The HN-GARCH model

The S. L. Heston and Nandi (2000) GARCH model is slightly different compared to other GARCH models. The model as a whole is designed to reach a closed form solution and option valuation that is analytically feasible. The S. L. Heston and Nandi (2000) model is defined as

$$r_t = r_f + \lambda h_t + \sqrt{h_t} z_t \tag{1}$$

$$h_t = \omega + b h_{t-1} + a(z_{t-1} - \gamma\sqrt{h_{t-1}})^2, \tag{2}$$

where r_t is the log return calculated as $ln\frac{S_t}{S_{t-1}}$ and S_t is the spot price at time t. r_f is the riskfree rate and h_t the conditional variance (conditioned on the information set I_{t-1}, which contains all available information up to time $t-1$) of the log return at time t. z_t is to be assumed as a standard normal error. λ, ω, b, a and γ are parameters that need to be estimated (in this case by maximum likelihood methods). λ corresponds to a variance risk premium and hence the return at time t depends on the actual weighted level of conditional variance. b expresses the influence of past conditional variance and a gives information about the kurtosis of the log returns' distribution. Last, γ controls the asymmetry of shocks and captures the so-called leverage effect found by Black (1976) and Christie (1982). Note, that in the case when $\gamma = \lambda = 0$, the shocks appear to be symmetricly distributed. If γ is positive, the distribution is negatively skewed, else we would observe a positive skewed distribution. To see this, the correlation between lagged conditional variance and the spot return is given by

$$Cov_{t-1}(h_{t+1}, log(S_t)) = -2a\gamma h_t. \tag{3}$$

With regard to valueing derivatives, riskneutral valuation principles become an issue. In a riskneutral world, investors do not require a higher rate of return to offset a potential higher level of risk. A riskneutral world possesses two properties, namely (1) the expected return for an investment equals the riskfree rate and (2) that riskfree rate is used to discount expected cashflows coming from an investment (e.g. an option) (see (Hull, 2009, p. 330)).

By some algebraic arrangements, S. L. Heston and Nandi (2000) are able to rewrite a risk neutral version of the equation system determined in equations 1 and 2[7]:

$$
\begin{aligned}
r_t &= r_f - \frac{1}{2}h_t + \sqrt{h_t}z_t^{rn} && (4)\\
h_t &= \omega + bh_{t-1} + a(z_{t-1}^{rn} - \gamma^{rn}\sqrt{h_{t-1}})^2 && (5)\\
z_t^{rn} &= z_t + (\lambda + \frac{1}{2})\sqrt{h_t}\\
\gamma^{rn} &= \gamma + \lambda + \frac{1}{2}
\end{aligned}
$$

Note, that for the risk neutral process in 4 and 5, λ is replaced by $\lambda^{rn} = -\frac{1}{2}$ and γ by γ^{rn}. Altogether, with this modification the one period return in equation (4) becomes the riskfree rate.

In order to estimate the model, we have to maximize the likelihood function. Since we know that z_t is standard normal, we automatically know that $E(r_t|I_{t-1}) = r_f + \lambda h_t$ and $Var(r_t|I_{t-1}) = \sqrt{h_t}^2 = h_t$. The likelihood function is given by

$$L(\theta) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi h_t}} exp\Big(-\frac{1}{2}\Big(\frac{r_t - r_f - \lambda h_t}{\sqrt{h_t}}\Big)^2\Big) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi h_t}} exp\Big(-\frac{1}{2}z_t^2\Big) \quad (6)$$

and the log likelihood function (without the constants) is given by

$$logL(\theta) = \sum_{t=1}^{T} -\frac{1}{2}lnh_t - z_t^2, \quad (7)$$

where θ contains all parameters that need to be estimated.

3.2. The HN option pricing formula

The value of a call option c is nothing else than the discounted expected value (expected today) of the amount the option is in the money at expiration day T. That is, a call option has value if and only if the stock price of the underlying at expiration date is higher than the strike price

[7] Note, that S. L. Heston and Nandi (2000) derived both equation systems (1, 2 and 4, 5) for the general case, e.g. $p, q \geq 1$, where p and q is the order of the GARCH model.

K. Mathematically this becomes

$$c = e^{(-r_f(T-t))} E_t(max(S_T - K, 0)), \tag{8}$$

whereas the expression $e^{(-r_f(T-t))}$ is the continuous compounded discount factor and $(T - t)$ is the number of days until expiration.

Hence, the question arises to evaluate the conditional expectation $E_t(\cdot)$. S. L. Heston and Nandi (2000) solved this problem by applying the theory of conditional moment generating function or conditional characteristic function, respectively and defined the conditional moment generating function of $x_T = log(S_T) \Leftrightarrow exp(x_T) = S_T$ at time T as

$$M_t(\phi) = E_t(S_T^{\phi}) = E_t(e^{\phi x_T}). \tag{9}$$

S. L. Heston and Nandi (2000) derive the expression for the moment generating function $M_t(\phi)$ as follows[8]

$$\begin{aligned} M_t(\phi) &= e^{\phi x_t + A_t + B_t h_{t+1}} \\ &= e^{\phi x_t} e^{A_t + B_t h_{t+1}} = S_t^{\phi} e^{A_t + B_t h_{t+1}}, \end{aligned} \tag{10}$$

whereas

$$A_t = A_{t+1} + \phi r_f + B_{t+1}\omega - 0.5ln(1 - 2aB_{t+1}) \tag{11}$$

$$B_t = \phi(\gamma + \lambda) - 0.5\gamma^2 + bB_{t+1} + 0.5\frac{(\phi - \gamma)^2}{1 - 2aB_{t+1}} \tag{12}$$

and the terminal conditions

$$A_T = B_T = 0 \tag{13}$$

The coefficients A_t and B_t can be computed recursively from the terminal conditions in 13[9]. When replacing ϕ by $i\phi$ the conditional moment generating function becomes the conditional characteristic function $\varphi_t(\phi)$. Moreover, by replacing $\lambda = -\frac{1}{2}$ and $\gamma = \gamma^{rn}$ we derive the risk neutral version of $\varphi_t(\phi)$, $\varphi_t^{rn}(\phi)$.

[8] They started their derivation by a guess that the moment generating function has a log-linear form and solved the equation following (Ingersoll, 1987, p. 397).

[9] For example, the value for $t = T - 1$ is given by: $A(T-1, T, i\phi) = A(T, T, i\phi) + i\phi r_f + B(T, T, i\phi)\omega - \frac{1}{2}ln(1 - 2aB(T, T, i\phi)) = i\phi r_f$ and $B(T-1, T, i\phi) = i\phi(\lambda+\gamma) - \frac{1}{2}\gamma^2 + bB(T, T, i\phi) + 0.5((i\phi)^2) - 2i\phi\gamma + \gamma^2 = i\phi\lambda + \frac{1}{2}i\phi i\phi$

Next, we evaluate the expectation in 8. It is well known that it holds that

$$E_t(max(e^{x_T} - K, 0)) = \int_{lnK}^{\infty} e^{x_T} f_t(x_T)\ dx_T - \int_{lnK}^{\infty} K f_t(x_T)\ dx_T \tag{14}$$

whereas $f_t(x_T)$ is the conditional density function of x_T. By some re-arrangement and applying fourier transform of the density, we receive the final option valuation formula

$$\begin{aligned} c &= e^{-r_f(T-t)} E_t^{rn}(max(S_T - K, 0)) = \\ &= \frac{S_t}{2} + \frac{e^{-r_f(T-t)}}{\pi} \int_0^{\infty} \Re\left[\frac{e^{-i\phi lnK} \varphi^{rn}(\phi+1)}{i\phi}\right] d\phi - \frac{Ke^{-r_f(T-t)}}{2} - \\ &- \frac{Ke^{-r_f(T-t)}}{\pi} \int_0^{\infty} \Re\left[\frac{e^{-i\phi lnK} \varphi^{rn}(\phi)}{i\phi}\right] d\phi. \end{aligned} \tag{15}$$

In (15) we used the Inversion Theorem introduced by Lévy (1925) and a corresponding alternative presentation found by Gil-Pelaez (1951). The detailed explanation and derivation of the formula is given in appendix A.

The riskneutral conditional characteristic function $\varphi_t^{rn}(\phi)$ is a function of A_t^{rn} and B_t^{rn}, whereas the coefficients are calculated as in (11) and (12) except λ and γ are replaced by their riskneutral versions λ^{rn} and γ^{rn}. Furthermore, the function depends on the conditional variance h_{t+1} (see equation 10) and therefore on the path of the underlying, which is the main difference to the Black and Scholes (1973) formula.

The forecast conditional variance can easily be computed by solving equation 1 for z_t and inserting it in equation 2. This manipulation leads directly to

$$h_{t+1} = \omega + bh_t + a\frac{(r_t - r_f - (\lambda+\gamma)h_t)^2}{h_t} \tag{16}$$

4. Empirical results

Firstly, an analysis of the GSCI spot index is provided by applying the HN GARCH(1,1) model. In a second step, I calculated option prices and option price paths based on the HN GARCH (1,1) model and compared them with the traditional Black-Scholes option pricing methodology. Prices derived from Monte Carlo simulation serve as a benchmark to compare the two methodologies.

4.1. Estimation results of the HN GARCH model

The first part of the analysis is about the application of the HN GARCH(1,1) model to the complete data set. The riskfree rate r_{rf} is the mean of the 3-months and-6 months treasury bill benchmark of Thomson Reuters[10] for a time period covering the entire sample or each subsample, respectively. Table 2 shows the estimation result of the HN GARCH(1,1) model for

	$r_{6M} = 1.36e-05$	$r_{6M} = 1.36e-05$ ($\gamma = 0$)	$r_{3M} = 9.67e-06$	$r_{3M} = 9.67e-06$ ($\gamma = 0$)
λ	-7.2511e-01	-7.3226e-01	-6.9499e-01	-7.0996e-01
ω	7.8652e-11	3.5835e-09	8.6571e-11	3.5907e-09
a	7.9398e-06	7.9150e-06	7.9266e-06	7.9168e-06
b	9.5632e-01	9.5642e-01	9.5638e-01	9.5641e-01
γ	1.8311e-01	0	2.3378e-01	0
θ	0.2140	0.2140	0.2140	0.2140
persistence	0.9563	0.9564	0.9564	0.9564
LL	9019.4550	9019.4320	9019.4670	9019.4430

Table 2: Heston Nandi GARCH(1,1) estimation output for 3-months and 6-months riskfree rate. θ is the annualized volatility calculated as $\theta = (252 \cdot \frac{\omega+a}{1-b-a\gamma})^{0.5}$ and the persistence parameter is computed as $b + a\gamma^2$.

the entire sample size. I estimated the model four times, changing the riskfree rate and the parameter restriction that γ (asymmetric parameter) equals zero.

Comparing the 6-months and 3-months riskfree rate, it seems that there is no difference in the estimation output. Furthermore, the log likelihood values are almost equal, as well as the persistence of the model and the annualized volatility θ. But surprisingly, the estimate for the risk premium λ is negative in all four cases. Hence, higher risk (in terms of conditional variance) leads to lower returns. Nevertheless, this can also be found in the literature, e.g. in Prokopczuk and Wese Simen (2014).

A more differentiated picture is painted when considering table 3, which reports the estimation results of the two sub-samples. The first time period was characterised by three major food crises, whereas the second time period covers a rather relaxing time frame. The estimates here show an absolutely different result.

For the time period in sub-sample 1, we observed a positive risk premium (λ). This could be due to the fact of financial turmoils/crises in food markets. Furthermore, the asymmetry parameter γ is also positive, indicating a negative correlation between returns and lagged variance. All in

[10] RIC: US3MT=RR

Panel 1: Sub-sample 1 from 1.1.2008 until 20.8.2012

	$r_{6M} = 1.95e-05$	$r_{6M} = 1.95e-05$ ($\gamma = 0$)	$r_{3M} = 1.48e-05$	$r_{3M} = 1.48e-05$ ($\gamma = 0$)
λ	9.2554e-01	1.4190e+00	7.6736e-01	1.4427e+00
ω	1.9071e-07	8.3767e-09	2.0804e-08	1.0725e-08
a	1.7978e-05	1.4601e-05	1.7961e-05	1.4600e-05
b	9.3102e-01	9.4990e-01	9.2915e-01	9.4990e-01
γ	1.9781e+01	0	2.3137e+01	0
θ	0.2719	0.2711	0.2720	0.2711
persistence	0.9381	0.9500	0.9388	0.9499
LL	4342.5320	4339.741	4342.6200	4339.743

Panel 2: Sub-sample 2 from 21.8.2012 until 31.12.2016

	$r_{6M} = 7.25e-06$	$r_{6M} = 7.25e-06$ ($\gamma = 0$)	$r_{3M} = 4.21e-06$	$r_{3M} = 4.21e-06$ ($\gamma = 0$)
λ	-6.2310e+00	-5.6383e+00	-6.2053e+00	-5.6175e+00
ω	1.3089e-08	7.2936e-09	1.5297e-08	9.3415e-09
a	3.7336e-06	3.7631e-06	3.7350e-06	3.7601e-06
b	9.5983e-01	9.5985e-01	9.5986e-01	9.5984e-01
γ	-6.0337e+00	0	-5.9133e+00	0
θ	0.1536	0.1538	0.1537	0.1538
persistence	0.9600	0.9600	0.9560	0.9600
LL	4723.154	4722.9910	4723.1510	4722.9920

Table 3: Heston Nandi GARCH(1,1) estimation output for 3-months and 6-months riskfree rate for two sub-samples (The break-date was estimated by change point analysis). θ is the annualized volatility calculated as $\theta = (252 \cdot \frac{\omega+a}{1-b-a\gamma})^{0.5}$ and the persistence parameter is computed as $b + a\gamma^2$.

all, the results for sub-sample 1 seem to be plausible. This point of view changes and the opposite picture can be drawn when analyzing the regression output of the second sub-sample. Here, both the risk premium λ and the asymmetry parameter γ turn to be negative. A negative estimate for γ indicates a positive skewed distribution, which was already shown in the descriptive analysis. Therefore, the model can capture the specific properties of each time series. It seems that the agricultural spot index behaves quite differently from other markets like equity markets what was already sugested in the descriptive analysis and is now confirmed.

To sum up the results, it seems that the behaviour of markets for agricultural products are divided into two sub-sets with complete different characteristics. The first sub-sample which covers major food crises is embossed by a positive risk premium and an asymmetric influence of shocks. The second sub-sample exhibits positive skewness and a negative variance premium.

Figure 2 shows in the upper part the estimated conditional volatility and in the lower part the conditional covariance between volatility and index log returns[11]. As already pointed out, con-

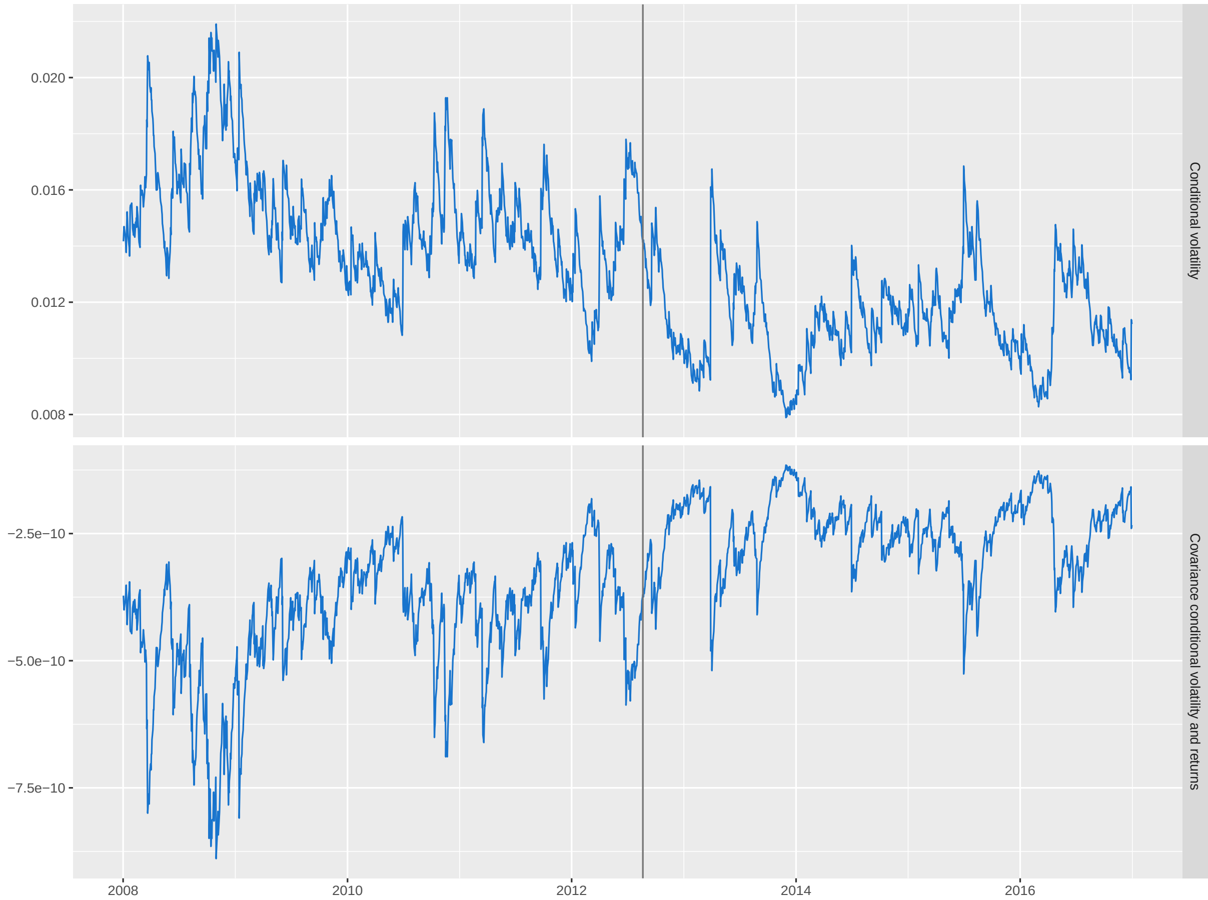

Figure 2: Conditional volatility and covariance between conditional variance in period t and returns in period t-1 as described in equation 3

ditional volatility is reducing since 2012 with contemporaneously increasing covariance between lagged conditional variance and log returns as described in equation 3.

4.2. HN GARCH option pricing

Next, I consecrate oneself to the calculation of option prices by applying the S. L. Heston and Nandi (2000) GARCH option pricing model.

4.2.1. Comparison of the HN option price, BS option price, and Monte Carlo simulation

I randomly chose two initial dates (19.05.2011 and 24.3.2014) in each sub-sample and based on this information, I calculated call prices. Each hypothetical option assumes 90 days to maturity and I set the strike price equal to the spot price at each starting date. This is a standard procedure when comparing option prices and can also be found in e.g. Byun (2011).

[11] For calculations, the estimates of the entire sample period were used.

Table 4 shows the results for each price computation and opposes S. L. Heston and Nandi (2000) (HN) and Black and Scholes (1973) (BS). The left side of the table prints the outcome for the high volatility time frame, whereas the right column shows the result for the low volatility period. In the high volatility sub-sample, the S. L. Heston and Nandi (2000) option model calculates a

	21.03. - 22.07.2011 S = K = 523.94			24.03. - 25.07.2014 S = K = 404.2		
	HN	BS	diff	HN	BS	diff
Sub-sample 1 (2008 - 2012)						
call price	29.9613	27.3220	2.6393			
Sub-sample 2 (2012 - 2016)						
call price				23.5209	23.6554	-0.1345

Table 4: Comparison of S. L. Heston and Nandi (2000)-GARCH option prices (HN) with Black and Scholes (1973) option prices (BS). S is the spot price at each starting date and K is the strike price. Diff is the difference between the calculated prices.

call price of 29.96 USD, whereas the Black and Scholes (1973) model returns a price of 27.32 USD. Therefore the Black and Scholes (1973) model underestimates the call price compared to the S. L. Heston and Nandi (2000) based price and it seems that the latter price model can handle high volatility in a better way, because presumably it is able to price the higher risk. Turning to the low volatility period the Black and Scholes (1973) model computes the higher call price (23.66 USD vs. 23.52 USD). Here, the Black and Scholes (1973) model overestimates the call price compared to the S. L. Heston and Nandi (2000) model. It is also notable that the price difference is quite smaller in the low volatility period. Hence, we have a first indication that the two models behave quite different in different volatility regimes.

Of course, there is no reason to generalize this observation, because we do not know what happens if the strike price K changes or the days until maturity alter. For this reason, as a next step of the analysis, I computed a maturity-moneyness-grid in table 4.2.1. Moneyness is defined as $\frac{S}{K}$ and goes stepwise from 0.90 to 1.05 or in other words, I discovered analytical at-the-money, in-the-money and out-of-the-money call prices and investigated how they differ when derived from Black and Scholes (1973) and S. L. Heston and Nandi (2000). Additionally, I varied time to maturity (in days) from 1 day left until 120 days left. As a result, the mean value of each grid cell is computed and presented in table 4.2.1.

This procedure is executed for the S. L. Heston and Nandi (2000) option price model, Black and Scholes (1973) option price model, and the price given by a Monte Carlo simulation used as a benchmark. Analyzing table 4.2.1 provides insights into the pricing behaviour in each volatility

	High volatility period S = 523.94			**Low volatility period** S = 404.20		
	<30	30 ≤ t ≤ 90	90< t ≤ 120	<30	30 ≤ t ≤ 90	90< t ≤ 120
0.90 ≤ $\frac{S}{K}$ ≤ 0.98						
HN	4.18	14.26	22.20	1.67	8.43	14.62
BS	2.59	11.73	19.86	2.00	9.05	15.40
Diff	1.59	2.53	2.34	-0.33	-0.62	-0.58
MC	3.50	14.55	23.99	1.56	7.58	13.25
0.98 < $\frac{S}{K}$ ≤ 1						
HN	11.56	24.96	33.93	6.23	16.33	23.79
BS	8.94	22.18	31.66	6.90	17.11	24.43
Diff	2.62	2.78	2.27	-0.67	-0.78	-0.64
MC	10.45	25.30	35.81	6.09	15.43	22.19
1 < $\frac{S}{K}$ ≤ 1.02						
HN	16.81	30.29	39.32	10.36	20.53	28.02
BS	14.27	27.60	37.12	11.00	21.29	28.64
Diff	2.54	2.69	2.20	-0.64	-0.76	-0.62
MC	15.72	30.63	41.16	10.23	19.67	26.49
1.02 < $\frac{S}{K}$ ≤ 1.05						
HN	24.99	37.62	46.48	17.23	26.43	33.69
BS	22.97	35.14	44.40	17.72	27.11	34.25
Diff	2.02	2.48	2.08	-0.49	-0.68	-0.56
MC	24.15	37.94	48.17	17.12	25.64	32.26
1.05 < $\frac{S}{K}$ ≤ 1.08						
HN	34.94	44.98	53.63	26.70	34.22	40.95
BS	34.94	45.06	53.73	26.95	34.76	41.45
Diff	0.00	-0.08	-0.10	-0.25	-0.53	-0.50
MC	35.68	47.46	57.14	26.60	33.54	39.71

Table 5: Comparison of S. L. Heston and Nandi (2000) (HN) call option prices, and Black and Scholes (1973) (BS) call option prices and Monte Carlo (MC) call prices in different volatility regimes, sorted by moneyness and time to maturity. Moneyness is calculated as $\frac{S}{K}$, whereas S is the spot price, and K the strike price.

regime of the analytical option prices. On the left side, the grid for the high volaility period is printed and assumes that the option is written on 21.03.2011 with an index price of 523.94 USD. On the right side, I demonstrated results of the low volatlity period with the initial date 24.03.2011 and an index of 404.20 USD.

In general, the pricing difference between the S. L. Heston and Nandi (2000) model and the Black and Scholes (1973) model is highest for options with time to maturity between 30 and 90 days. This observation can be made for the high volatility period as well as for the low volatility regime. Moreover, the pricing error between the two models is quite larger in the high volatility period compared to the low volatility period. This is due to the fact that the Black and Scholes (1973) model does not account for time varying volatility and therefore, in times of small price movements, the assumption of a constant variance is not that violated as it is in the high volatility period with wide price fluctuations. Thus, this result makes sense economically and mathematically and satisfies the expectation.

In times of *high conditional volatility* the Black and Scholes (1973), price is always lower than the S. L. Heston and Nandi (2000) price for each time to maturity and moneyness combination and the above given explanation for this observation can thus be confirmed. The result when comparing the pricing error between the S. L. Heston and Nandi (2000) model and the Monte Carlo simulation, and between the Black and Scholes (1973) and the Monte Carlo simulation, is that the latter systematically understates the MC option price in the high volatility period. The difference of the analytical prices is highest for options with moneyness between 0.98 and 1 and therefore at-the-money call options. This can be explained trough the fact that especially for near-the-money options volatiliy plays an important role, because the higher the volatility the higher the likelihood that the option runs in-the-money. The almost same observation can be made for options with moneyness between 1 and 1.02 (at- or in-the-money options) and the deeper the option is in-the-money the less is the analytical pricing error.

Taking into account all these aspects, it turns out that the Black and Scholes (1973) price performs poor compared to the S. L. Heston and Nandi (2000) model when using the Monte Carlo price as a benchmark. The S. L. Heston and Nandi (2000) model does way better in all combinations of moneyness and time to maturity and appears more stable. However, this is what was expected because of the fact of ignoring time varying variance by the Black and Scholes (1973) model.

In the second period with rather *low volatility*, the Black and Scholes (1973) model always overstates both the Monte Carlo option price and the S. L. Heston and Nandi (2000) result. Also in this volatility case it can be observed that the pricing difference is highest for options with maturity between 30 and 90 days. This valuation error decreases with increasing moneyness. Hence, there are some basic parallels in pricing behaviour.

Altough, the price that results from the S. L. Heston and Nandi (2000) model also overstates the Monte Carlo simulated price of a call option, the pricing error is much smaller over different days to maturity and moneyness. Moreover, the Black and Scholes (1973) price is almost always higher than the S. L. Heston and Nandi (2000) price. This is, again, due to the fact that Black and Scholes (1973) assumes a constant variance over time that is likely too high compared to the real (lower) volatility in this time period.

Comparing the prices in each period as a whole the result shows that the pricing differences are higher in the high volatility period and way smaller in the low volatility period. This observation leads to the conclusion that the state of volatiliy plays a major role in choosing an appropriate pricing model. Moreover it seems that the S. L. Heston and Nandi (2000) pricing model is the better choice especially in high volatility periods.

In a next step, I calculated option prices from BS and HN by updating the model parameters each point in time. The resulting rolling window option price is then investigated.

4.2.2. Rolling window option price calculation

I am now turning to a rolling window analysis to explore the evolution of analytical prices during a certain time frame of a 90-day-to-maturity-option. Hence, I estimated the S. L. Heston and Nandi (2000) model 90 times for each option, always using an updated information set. With the updated parameters, I calculated the new option price with a decreasing time to maturity. Finally, I compared the S. L. Heston and Nandi (2000) option model with the results of the Black and Scholes (1973) option model. As in the analysis above, I also calculated mean values of different moneyness factors to not only consider one strike, but rather a bulk of different strikes. Furthermore, I explored two different sets of moneyness regions seperatly. The outcome is plotted in figure 3 for moneyness between 0.95 and 1 and in figure 4 for moneyness between 1 and 1.05. The upper part in each figure shows the analytical price paths for the options derived from the S. L. Heston and Nandi (2000) model (blue) and from the Black and Scholes (1973) model (black). The lower part of each plot shows the time varying price difference calculated as the Heston Nandi price minus the Black and Scholes price. Lastly, the left part of each figure corresponds to the high volatility period and the right part to the low volatility period.

I started the analysis with the left side (high volatility period from 2008 until 2012) of figure 3. The mean price difference of initally out of the money options is positive in the beginning of the option life, indicating that S. L. Heston and Nandi (2000) prices options with a higher value compared to Black and Scholes (1973). After around ten days, the sign changes and becomes

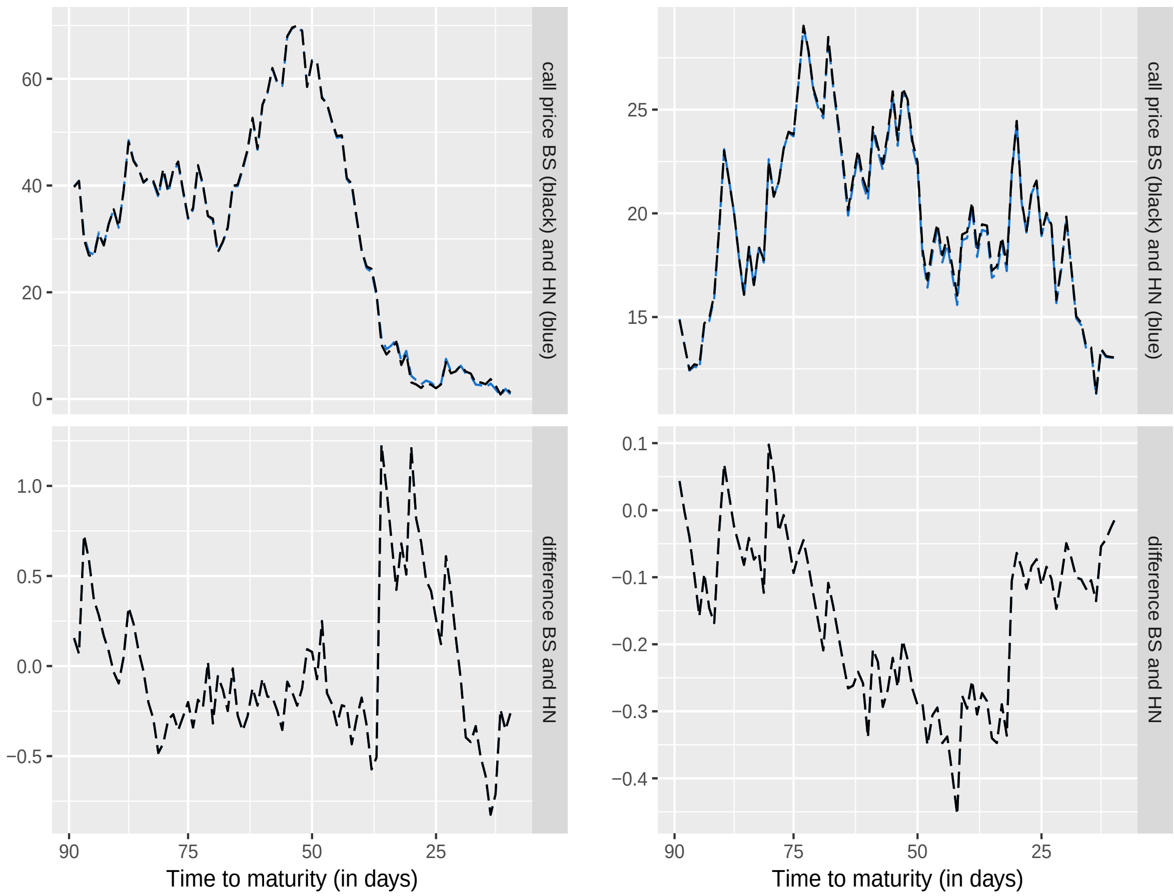

Figure 3: Call prices of the S. L. Heston and Nandi (2000) (HN) model and the Black and Scholes (1973) (BS) model (upper part) compared to the error (lower part) for two hypothetical call price paths. (left, high volatility and right, low volatility). Moneyness between 0.95 and 1.

negative. This is due to the fact that the volatility is shrinking (not shown) exactly in this time frame within the defined high volatility period where the sign of the pricing error is negative.

On the right side of the figure (low volatility period), the Black and Scholes (1973) prices the option higher for almost the whole time, since S. L. Heston and Nandi (2000) considers lower conditional volatility.

I am now turning to figure 4 with initial moneyness between 1 and 1.05 a quite similar pattern. The absolute option price differences in the upper part of the figure are due to different maturities. Beyond that, the relative pricing error (lower part of the figure) is closer to zero, but the reason for that is not the model behavior itself (because of similar paths), but rather different defined moneynesses factors.

Nevertheless, the suggestion that the S. L. Heston and Nandi (2000) option pricing model calculates more reliable call values in rough times (in terms of high volatility) can be verified when

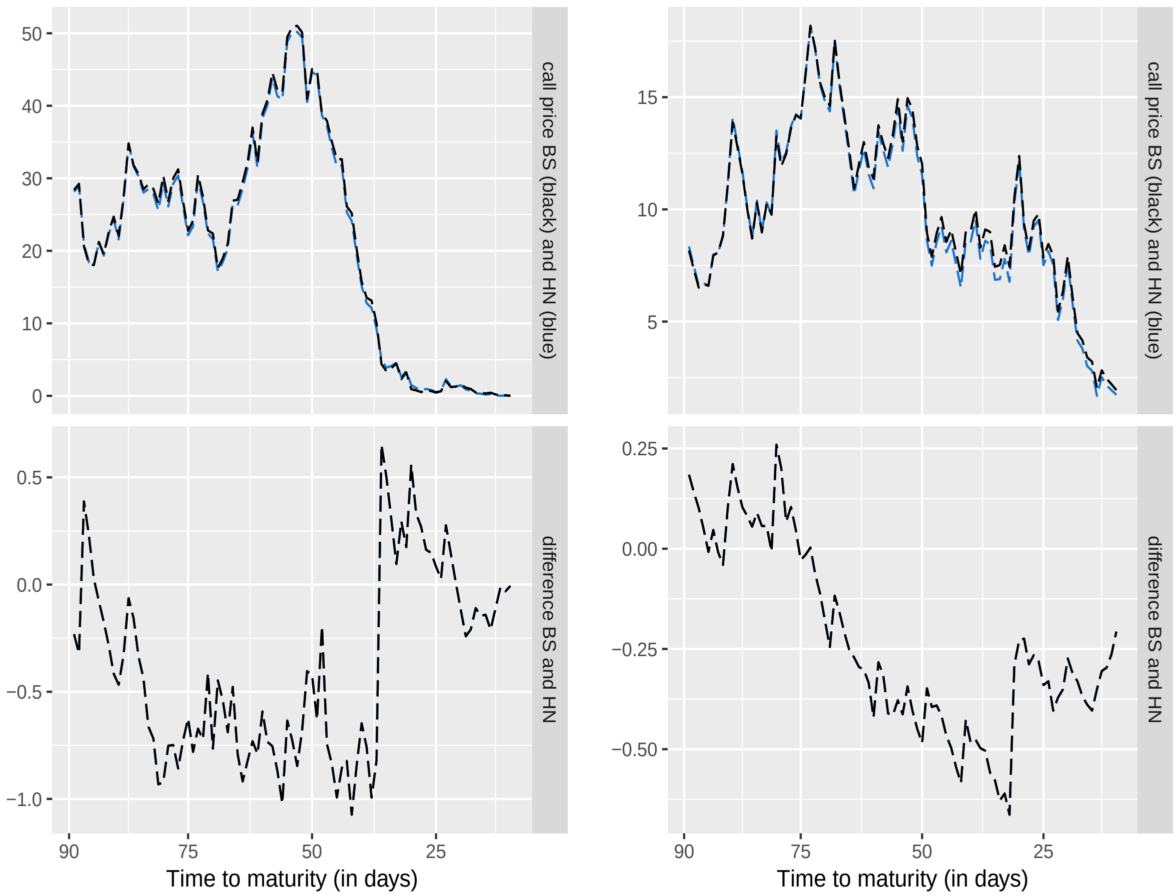

Figure 4: Call prices of the S. L. Heston and Nandi (2000) (HN) model and the Black and Scholes (1973) (BS) model (upper part) compared to the error (lower part) for two hypothetical call price paths. (left, high volatility and right, low volatility). Moneyness between 1 and 1.05.

looking at the root mean squared error (RMSE)[12].

For the high volatility period and moneyness between 0.95 and 1, the RMSE is calculated as 0.4097, whereas the root mean squared error for the low volatility period can be computed as 0.1968.

For moneyness between 1 and 1.05, the RMSE becomes 0.5741 in the high volatility frame vs. 0.3272 in the low volatility period.

This means that in high volatility times the difference between the call prices derived from the two models is much higher than in times with low volatility which can be traced back to the different volatility settings in each model. Additionally, it seems that the pricing error is also dependent on moneyness of the option. As for in-the-money options the RMSE is higher compared to out of money options in each volatility regime.

[12] The root mean squared error (RMSE) is calculated as $\sqrt{\sum_{i=1}^{n}(c_{HN}-c_{BS})^2/n}$, c_HN is the call price coming from the S. L. Heston and Nandi (2000) model and c_BS is the call price received from the Black and Scholes (1973) model.

Summarizing the above, the rolling window analysis gives similar conclusion as in 4.2.1 and discovers different pricing errors for the S. L. Heston and Nandi (2000) and Black and Scholes (1973) model between different moneyness assumptions.

5. Conclusion

This paper applies both the S. L. Heston and Nandi (2000) GARCH model and the S. L. Heston and Nandi (2000) option valuation framework to S&P GSCI Agricultural Spot Index returns and compares the call prices with the prices from the Black and Scholes (1973) option pricing model as well as with the results obtained from a Monte Carlo Simulation.

The analysis suggests, that the investigated time series can be divided into two sub-samples of high volatility (before 2012) and low volatility (after 2012). The two volatility regimes found by a breakpoint analysis are in line with the literature. In the case of the high volatility period, both risk premium and asymmetry tends to be positive, whereas the sign switched when investigating the low volatility period.

Moreover, the analysis gives interesting key insides of the pricing behaviour of S. L. Heston and Nandi (2000) GARCH options compared to Black and Scholes (1973) option prices. It turns out that the pricing error between the S. L. Heston and Nandi (2000) model and the Black and Scholes (1973) model behaves very different for both different maturities and moneyness combinations and heavily depends on the volatility regimes. This conclusion can be drawn from both a static option price calculation and a calculation within a rolling window framework.

A. Derivation of S. L. Heston and Nandi (2000) option valuation

Before we start, I would like to list some important relations we need at a later point of time.

- $log(S_T) = x_T \Leftrightarrow S_T = e^{x_T}$, where S_T is the terminal value of the underlying.
- $f_t(x_T)$ conditional density function of x_T at time t.
- $f_t^{adj}(x_T) = \frac{e^x f_t(x_T)}{M_t(1)}$ adjusted conditional density function of x_T at time t.
- $M_t(\phi) = E_t(e^{log(S_T)\phi}) = E_t(S_T^{\phi}) = E_t(e^{x_T\phi}) = \int_{-\infty}^{\infty} e^{\phi x_T} f_t(x_T) dx_T$ conditional moment generating function of $f_t(x_T)$ at time t.
- $\varphi_t(\phi)$ is the conditional characteristic function of $f_t(x_T)$ at time t.
- $M_t^{adj}(\phi) = \int_{-\infty}^{\infty} e^{\phi x_T} f_t^{adj}(x_T) dx_T = \int_{-\infty}^{\infty} e^{\phi x_T} \frac{e^{x_T} f_t(x_T)}{M_t(1)} dx_T = \frac{1}{M_t(1)} \int_{-\infty}^{\infty} e^{(\phi+1)x_T} f_t(x_T) dx_T = \frac{M_t(\phi+1)}{M_t(1)}$ conditional moment generating function of $f_t^{adj}(x_T)$ at time t.
- $\varphi_t^{adj}(\phi) = \frac{\varphi_t(\phi+1)}{\varphi_t(1)}$ conditional characteristic function of $f_t^{adj}(x_T)$ at time t.

First, we evaluate the conditional expectation that the option is in the money at expiration date using the information listed above.

$$\begin{aligned} E_t(max(e^{x_T} - K, 0)) &= \int_{lnK}^{\infty} e^{x_T} f_t(x_T)\ dx_T - \int_{lnK}^{\infty} K f_t(x_T)\ dx_T \\ &= \int_{lnK}^{\infty} e^{x_T} \frac{f_t^{adj}(x_T)}{e^{x_T}} \varphi_t(1)\ dx_T - K \int_{lnK}^{\infty} f_t(x_T)\ dx_T \\ &= \varphi_t(1) \underbrace{\int_{lnK}^{\infty} f_t^{adj}(x_T)\ dx_T}_{[1]} - K \underbrace{\int_{lnK}^{\infty} f_t(x_T)\ dx_T}_{[2]} \end{aligned} \tag{17}$$

With this algebraic arrangements we now have to calculate probabilites that the underlying stock price is greater than the strike price K at maturity. This is done by evaluating the integrals, which is nothing else than calculating probabilities by applying the inversion theorem introduced by Lévy (1925). The inversion theorem gives an unique relationship between the cumulative distribution function and the characteristic function. In this case, we resort to the presentation of the inversion theorem found by Gil-Pelaez (1951).

$$\begin{aligned}
[1] \quad \int_{lnK}^{\infty} f_t^{adj}(x_T)\, dx_T = P(x_T > lnK) &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-i\phi lnK}\varphi_t^{adj}(\phi)}{i\phi}\right) d\phi = \\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-i\phi lnK}\varphi_t(\phi+1)}{i\phi\varphi_t(1)}\right) d\phi \qquad (18)\\
[2] \quad \int_{lnK}^{\infty} f_t(x_T)\, dx_T = P(x_T > lnK) &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-i\phi lnK}\varphi_t(\phi)}{i\phi}\right) d\phi \qquad (19)
\end{aligned}$$

$$\begin{aligned}
E_t(max(e^{x_T} - K, 0)) = E_t(max(S_T - K, 0)) &= \varphi_t(1)\ [1] - K\ [2] = \\
&= \varphi_t(1)\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-i\phi lnK}\varphi_t(\phi+1)}{i\phi\varphi_t(1)}\right) d\phi\right) \\
&\quad - K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-i\phi lnK}\varphi_t(\phi)}{i\phi}\right) d\phi\right) \\
&= \frac{\varphi_t(1)}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-i\phi lnK}\varphi_t(\phi+1)}{i\phi}\right) d\phi \\
&\quad - K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-i\phi lnK}\varphi_t(\phi)}{i\phi}\right) d\phi\right)
\end{aligned} \qquad (20)$$

We can now insert the characteristic function given in section 3.2. The complete derivation of the characteristic function can be found in S. L. Heston and Nandi (2000). However, the characteristic function from the S. L. Heston and Nandi (2000) model is

$$\begin{aligned}
\varphi_t(\phi) = S_t^{i\phi} e^{A_t + B_t h_{t+1}} = e^{i\phi x_t + A_t + B_t h_{t+1}} &= e^{i\phi lnS_t + A_t + B_t h_{t+1}} \\
\varphi_t(\phi+1) &= e^{(i\phi+1) lnS_t + A_t + B_t h_{t+1}}.
\end{aligned} \qquad (21)$$

Inserting the explicit expression of the characteristic function in (20) and discounting with $e^{-r_f(T-t)}$ we receive

$$\begin{aligned}
c = \frac{S_t}{2} + \frac{e^{-r_f(T-t)}}{\pi}\int_0^{\infty} &\Re\left(\frac{e^{-i\phi lnK} e^{(i\phi+1) lnS_t + A_t + B_t h_{t+1}}}{i\phi}\right) d\phi \\
&- e^{-r_f(T-t)} K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-i\phi lnK} e^{i\phi lnS_t + A_t + B_t h_{t+1}}}{i\phi}\right) d\phi\right)
\end{aligned} \qquad (22)$$

Note that equation (22) is not the correct value of the option, since we did not consider riskneutrality . Rather we must consider the riskneutral coefficients.

$$A_t^{rn} = A_{t+1}^{rn} + i\phi r_f + B_{t+1}^{rn}\omega - 0.5ln(1 - 2aB_{t+1}^{rn}) \tag{23}$$

$$B_t^{rn} = i\phi(\gamma^* + \lambda^*) - 0.5(\gamma^*)^2 + bB_{t+1}^{rn} + 0.5\frac{(i\phi - \gamma^*)^2}{1 - 2aB_{t+1}^{rn}} \tag{24}$$

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